Journal of Statistical Physics, Vol. 106, Nos. 1/2, January 2002 (© 2002)

Large-Distance Behavior of Particle Correlations in the Two-Dimensional Two-Component Plasma

L. Šamaj[1,2] **and B. Jancovici**[1]

Received April 10, 2001; accepted August 10, 2001

The model under consideration is a two-dimensional two-component plasma, i.e., a continuous system of two species of pointlike particles of opposite charges ± 1, interacting through the logarithmic Coulomb interaction. Using the exact results for the form-factors of an equivalent Euclidean sine-Gordon theory, we derive the large-distance behavior of the pair correlation functions between charged particles. This asymptotic behavior is checked on a few lower orders of its β-expansion (β is the inverse temperature) around the Debye–Hückel limit $\beta \to 0$, and at the free-fermion point $\beta = 2$ at which the collapse of positive-negative pairs of charges occurs.

KEY WORDS: Coulomb systems; sine-Gordon model; exactly solvable models; correlation functions; form factors.

1. INTRODUCTION

We consider a two-dimensional (2D) two-component plasma (TCP), i.e., a neutral continuous system of two species of pointlike particles of opposite charges ± 1, interacting through the 2D logarithmic Coulomb interaction. Classical equilibrium statistical mechanics is used. The system is stable against collapse of positive-negative pairs of charges for the dimensionless coupling constant (inverse temperature) $\beta < 2$. In the region $\beta \geqslant 2$, one has to attach to particles a hard core in order to prevent the collapse. In the limit of a small, but nonzero, hard core, the Kosterlitz–Thouless phase

[1] Laboratoire de Physique Théorique, Université de Paris-Sud, Bâtiment 210, 91405 Orsay Cedex, France (Unité Mixte de Recherche no. 8627 - CNRS); e-mails: Ladislav.Samaj@th.u-psud.fr and Bernard.Jancovici@th.u-psud.fr

[2] On leave from the Institute of Physics, Slovak Academy of Sciences, Bratislava, Slovakia; e-mail: fyzimaes@savba.sk

0022-4715/02/0100-0301/0

transition from a high-temperature conducting phase to a low-temperature insulating phase[1] takes place at around $\beta = 4$.

The 2D TCP is the first continuous fluid in dimension higher than one with an exactly solvable thermodynamics, in the bulk as well as at the surface for specific boundary conditions. The bulk equation of state has been known for a very long time.[2] The other bulk thermodynamic properties (free energy, internal energy, specific heat, etc.) have been obtained in the whole stability range $\beta < 2$ of the plasma in a recent paper.[3] The mapping onto a bulk 2D Euclidean (classical) sine-Gordon field theory with a specific (conformal) normalization of the cos-field was made, and recent results about that field theory,[4–6] were used. In subsequent works, the surface tension of the same model in contact with an ideal conductor[7] and an ideal dielectric[8] rectilinear walls was obtained. A mapping onto a boundary 2D sine-Gordon field theory with a Dirichlet and a Neumann boundary condition, respectively, was made, and known results[9] about those integrable boundary field theories[10] were applied.

In this paper, we derive the large-distance behavior of the bulk particle correlation functions for the 2D TCP by exploring the form-factor theory of the equivalent sine-Gordon model.

Exact sum rules are known for the correlation functions. The zeroth and second moments of the charge correlation function have the well-known Stillinger–Lovett values.[11] The zeroth moment of the density correlation function is related to the compressibility (here exactly known), and the second moment has been recently obtained.[12, 13] An exact analysis of pair correlations in the 2D TCP can be done in two cases: in the high-temperature Debye–Hückel limit $\beta \to 0$ and just at the collapse point $\beta = 2$ which corresponds to the free-fermion point of an equivalent Thirring model (although, at a given fugacity, the free energy diverges, truncated particle distributions are finite at $\beta = 2$ and, as is generally believed, also in the whole range $\beta < 4$). We have not found in the literature an exhaustive treatment of the limit $\beta \to 0$, so we study this limit in detail in the present work by using the technique of a renormalized Mayer expansion in density.[13, 14] At $\beta = 2$, the exact forms of the bulk truncated distributions (Ursell functions) were found in ref. 15 via a continualization of the Gaudin model.[16] The Ursell functions were computed at $\beta = 2$ also for inhomogeneous situations when the 2D TCP, being confined to domains of various geometry, is in contact with a charged hard wall, polarizable interface,[17] an ideal conductor wall[17–19] and an ideal dielectric wall.[20, 21] A short-distance expansion of pair correlations can be done in principle for an arbitrary β by using Dotsenko's integrals;[22] interestingly, correlations between equally charged particles change their analytic structure at short distance at $\beta = 1$.[23]

In the 2D sine-Gordon theory, a differential equation for generalized correlation functions was derived at the free-fermion point.[24] The classical 2D sine-Gordon model can be regarded as a conformal (namely Gaussian) field theory, perturbed by the cos-field. For such theories, the short-distance expansion for multipoint correlation functions can be systematically obtained by using the Operator product expansion,[25] combined with the Conformal perturbation theory.[26] The most efficient way to study the large-distance behavior of correlations is provided by the form-factor approach. For the 2D sine-Gordon model, the form factors of some local operators for all kinds of particles the theory contains were evaluated in a series of papers.[27–30] In particular, the form factors for the lightest particle (elementary breather), dominant when this particle exists, are analytic continuations of their counterparts in the sinh-Gordon theory.[31, 32] In the sinh-Gordon model, the corresponding particle is the only massive one; the contributions of all form factors were summed up and a closed expression for any correlation function as a determinant of an integral operator was obtained in ref. 33.

The paper is organized as follows. In Section 2, we introduce the notation and briefly review some important information on the mapping of the 2D TCP onto the 2D sine-Gordon model.[3] Section 3 deals with the form-factor theory for the sine-Gordon model, written in a way accessible to non-specialists as well. The large-distance asymptotic of pair particle distributions for the 2D TCP is presented. The obtained results are checked around the Debye–Hückel limit $\beta \to 0$ and at the free fermion $\beta = 2$ point in Section 4. A brief recapitulation and some concluding remarks are given in Section 5.

2. SINE-GORDON REPRESENTATION OF THE PLASMA

We consider an infinite 2D space of points $\mathbf{r} \in R^2$. The 2D TCP, defined in this space, consists of point particles $\{j\}$ of charge $\{\sigma_j = \pm 1\}$, immersed in a homogeneous medium of dielectric constant $= 1$. The interaction energy of particles is given by $\sum_{i<j} \sigma_i \sigma_j v(|\mathbf{r}_i - \mathbf{r}_j|)$, where the Coulomb potential v is the solution of the 2D Poisson equation

$$\Delta v(\mathbf{r}) = -2\pi \delta(\mathbf{r}) \tag{2.1}$$

Explicitly, $v(\mathbf{r}) = -\ln(|\mathbf{r}|/r_0)$ where r_0 is a length scale set for simplicity to unity. We will work in the grand canonical ensemble, characterized by the inverse temperature β and the equivalent fugacities of the positively and negatively charged particles, $z_+ = z_- = z$. Due to the charge $\pm$ symmetry, the induced particle densities are $n_+ = n_- = n/2$ (n is the total number density of particles).

Using the fact that $-\Delta/(2\pi)$ is the inverse operator of $v(\boldsymbol{r})$ [see Eq. (2.1)], the grand partition function Ξ of the 2D TCP can be turned via the Hubbard–Stratonovitch transformation (see, e.g., review ref. 34) into

$$\Xi(z) = \frac{\int \mathcal{D}\phi \exp(-\mathcal{A}(z))}{\int \mathcal{D}\phi \exp(-\mathcal{A}(0))} \tag{2.2}$$

where

$$\mathcal{A}(z) = \int \mathrm{d}^2 r \left[\frac{1}{16\pi} (\nabla\phi)^2 - 2z \cos(b\phi) \right] \tag{2.3a}$$

$$b^2 = \beta/4 \tag{2.3b}$$

is the Euclidean action of the 2D classical sine-Gordon theory. Here, $\phi(\boldsymbol{r})$ is a real scalar field, $\int \mathcal{D}\phi$ denotes the functional integration over this field and the fugacity z is renormalized by a self-energy term. The sine-Gordon representation of the density of particles of one sign is

$$\begin{aligned} n_\sigma &= \left\langle \sum_j \delta_{\sigma,\sigma_j} \delta(\boldsymbol{r} - \boldsymbol{r}_j) \right\rangle \\ &= z_\sigma \langle \mathrm{e}^{\mathrm{i}\sigma b\phi} \rangle \end{aligned} \tag{2.4}$$

where $\langle \cdots \rangle$ denotes the averaging over the sine-Gordon action (2.3). The equality of densities $n_+ = n_-$ for the considered $z_+ = z_- = z$ is a special case of a general symmetry relation $\langle \mathrm{e}^{\mathrm{i}a\phi} \rangle = \langle \mathrm{e}^{-\mathrm{i}a\phi} \rangle$, a arbitrary, which results from the invariance of the sine-Gordon action (2.3) with respect to the transformation $\phi \to -\phi$. For two-body densities, one gets

$$\begin{aligned} n_{\sigma\sigma'}(\boldsymbol{r}, \boldsymbol{r}') &= \left\langle \sum_{j \neq k} \delta_{\sigma,\sigma_j} \delta(\boldsymbol{r} - \boldsymbol{r}_j)\, \delta_{\sigma',\sigma_k} \delta(\boldsymbol{r}' - \boldsymbol{r}_k) \right\rangle \\ &= z_\sigma z_{\sigma'} \langle \mathrm{e}^{\mathrm{i}\sigma b\phi(\boldsymbol{r})} \mathrm{e}^{\mathrm{i}\sigma' b\phi(\boldsymbol{r}')} \rangle \end{aligned} \tag{2.5}$$

Clearly, $n_{++}(\boldsymbol{r}, \boldsymbol{r}') = n_{--}(\boldsymbol{r}, \boldsymbol{r}')$ and $n_{+-}(\boldsymbol{r}, \boldsymbol{r}') = n_{-+}(\boldsymbol{r}, \boldsymbol{r}')$. For our purpose, it is useful to consider at the two-particle level also the pair distribution functions

$$g_{\sigma\sigma'}(\boldsymbol{r}, \boldsymbol{r}') = \frac{n_{\sigma\sigma'}(\boldsymbol{r}, \boldsymbol{r}')}{n_\sigma n_{\sigma'}} \tag{2.6}$$

the pair correlation functions

$$h_{\sigma\sigma'}(\boldsymbol{r}, \boldsymbol{r}') = g_{\sigma\sigma'}(\boldsymbol{r}, \boldsymbol{r}') - 1 \tag{2.7}$$

and the Ursell functions

$$U_{\sigma\sigma'}(\boldsymbol{r}, \boldsymbol{r}') = n_\sigma n_{\sigma'} h_{\sigma\sigma'}(\boldsymbol{r}, \boldsymbol{r}') \tag{2.8}$$

(denoted as $\rho^{(2)T}_{\sigma\sigma'}$ in refs. 15 and 17).

The parameter z in (2.3), i.e., the fugacity renormalized by a diverging self-energy term, gets a precise meaning when one fixes the normalization of the cos-field. In particular, it was shown for the 2D TCP,[23] that the behavior of the two-body densities for oppositely charged particles is dominated at short distance by the Boltzmann factor of the Coulomb potential,

$$n_{+-}(\boldsymbol{r}, \boldsymbol{r}') \sim z_+ z_- \, |\boldsymbol{r} - \boldsymbol{r}'|^{-\beta} \qquad \text{as} \quad |\boldsymbol{r} - \boldsymbol{r}'| \to 0 \tag{2.9}$$

With respect to (2.5) and the definition (2.3b), one has in the sine-Gordon picture

$$\langle e^{ib\phi(r)} e^{-ib\phi(r')} \rangle \sim |\boldsymbol{r} - \boldsymbol{r}'|^{-4b^2} \qquad \text{as} \quad |\boldsymbol{r} - \boldsymbol{r}'| \to 0 \tag{2.10}$$

This short-distance normalization of the exponential fields, under which the diverging self-energy factor disappears from the statistical relationships, makes the mapping complete. In quantum field theory, (2.10) is known as the conformal normalization, and it corresponds to taking (2.3) as a Gaussian conformal field theory perturbed by the relevant operator $\cos(b\phi)$. The conformal normalization makes the bridge between the 2D TCP and the sine-Gordon model, and allows one to transfer results from one theory to the other.[3]

Let us now summarize the known facts about the sine-Gordon theory (2.3). The theory has a discrete symmetry $\phi \to \phi + 2\pi j/b$ (j any integer). This symmetry is spontaneously broken, and the sine-Gordon theory is massive, in the region $0 < b^2 < 1$ (i.e., $0 < \beta < 4$ for the 2D TCP). In this region, one has to consider one of infinitely many ground states $\{|0_j\rangle\}$ characterized by $\langle \phi \rangle_j = 2\pi j/b$, say that with $j = 0$. The 2D classical sine-Gordon theory is integrable.[35] Its particle spectrum consists of one soliton-antisoliton pair (A_+, A_-) (the corresponding particle index $\epsilon = \pm$ denotes the topological $U(1)$ charge) with equal soliton and antisoliton masses $m_+ = m_- = M$ and of soliton-antisoliton bound states ("breathers") $\{B_j; j = 1, 2, \ldots < 1/\xi\}$ (the corresponding particle index $\epsilon = j$), whose number at a given b^2 depends on the inverse of the parameter

$$\xi = \frac{b^2}{1-b^2} \qquad \left(= \frac{\beta}{4-\beta} \right) \tag{2.11}$$

The range $0 < \xi < 1$ ($0 < \beta < 2$) is an attractive one (breathers exist), the range $1 \leqslant \xi < \infty$ ($2 \leqslant \beta < 4$) is a repulsive one (breathers do not exist); $\xi = 1$ ($\beta = 2$) corresponds to the free-fermion (collapse) point. The masses of breathers $\{B_j\}$ are given by the formula

$$m_j = 2M \sin\left(\frac{\pi\xi}{2} j \right) \tag{2.12}$$

The lightest breather B_1 is usually called the elementary one. The energy E and the momentum p of a particle ϵ of mass m_ϵ are parametrized by the rapidity $\theta \in (-\infty, \infty)$ as follows

$$E = m_\epsilon \cosh\theta, \qquad p = m_\epsilon \sinh\theta \tag{2.13}$$

Like in other 2D integrable field theories, the N-particle scattering amplitudes are purely elastic (the number of particles is conserved, the incoming and outgoing momenta are the same) and factorized into $N(N-1)/2$ two-particle S-matrices which are determined exactly by exploring their general properties: unitarity and crossing symmetry, validity of the Yang–Baxter equation and the assumption of "maximal analyticity."

The dimensionless specific grand potential ω of the sine-Gordon theory (2.2), (2.3), defined by

$$-\omega = \lim_{V\to\infty} \frac{1}{V} \ln \Xi \tag{2.14}$$

was found in ref. 4 by using the Thermodynamic Bethe ansatz:

$$-\omega = \frac{m_1^2}{8\sin(\pi\xi)} \tag{2.15}$$

Under the conformal normalization of the exponential fields (2.10), the relationship between the parameter z and the soliton mass M was established in ref. 5:

$$z = \frac{\Gamma(b^2)}{\pi\Gamma(1-b^2)} \left[M \frac{\sqrt{\pi}\,\Gamma((1+\xi)/2)}{2\Gamma(\xi/2)} \right]^{2-2b^2} \tag{2.16}$$

where Γ stands for the Gamma function. As a result,

$$\langle e^{ib\phi} \rangle = \frac{1}{2} \frac{\partial(-\omega)}{\partial z} = \frac{M^2}{8z(1-b^2)} \tan\left(\frac{\pi\xi}{2}\right) \tag{2.17}$$

Relations (2.14)–(2.17), together with the equality

$$n = 2z\langle e^{ib\phi} \rangle \tag{2.18}$$

constitute the complete set of equations for determining the thermodynamics of the 2D TCP.[3] They will be extensively used in the following section.

3. FORM-FACTOR THEORY FOR THE 2D TCP

In a 2D integrable system with particle spectrum $\{\epsilon\}$, correlation functions of local operators $\mathcal{O}_a$ (a is a free parameter) can be written as an infinite convergent series over multi-particle intermediate states. For the two-point correlation function, one has

$$\langle \mathcal{O}_a(\mathbf{r})\, \mathcal{O}_{a'}(\mathbf{r}') \rangle = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\epsilon_1,\dots,\epsilon_N} \int_{-\infty}^{\infty} \frac{d\theta_1 \cdots d\theta_N}{(2\pi)^N} F_a(\theta_1,\dots,\theta_N)_{\epsilon_1\cdots\epsilon_N}$$
$$\times\, {}^{\epsilon_N\cdots\epsilon_1}F_{a'}(\theta_N,\dots,\theta_1) \exp\left(-|\mathbf{r}-\mathbf{r}'| \sum_{j=1}^{N} m_{\epsilon_j} \cosh\theta_j\right) \tag{3.1}$$

The first $N=0$ term of the series is nothing but the decoupling $\langle \mathcal{O}_a \rangle \langle \mathcal{O}_{a'} \rangle$. The form factors

$$F_a(\theta_1,\dots,\theta_N)_{\epsilon_1\cdots\epsilon_N} = \langle 0|\, \mathcal{O}_a(\mathbf{0})\, |Z_{\epsilon_1}(\theta_1),\dots, Z_{\epsilon_N}(\theta_N)\rangle \tag{3.2a}$$

$${}^{\epsilon_N\cdots\epsilon_1}F_{a'}(\theta_N,\dots,\theta_1) = \langle Z_{\epsilon_N}(\theta_N),\dots, Z_{\epsilon_1}(\theta_1)|\, \mathcal{O}_{a'}(\mathbf{0})\, |0\rangle \tag{3.2b}$$

are the matrix elements of the operator at the origin, between an N-particle in-state (being a linear superposition of free one-particle states $|Z_\epsilon(\theta)\rangle$) and the vacuum. They depend on particle rapidities only through their differences $\theta_{jk} = \theta_j - \theta_k$. The form factors can be determined exactly (for the underlying sine-Gordon theory, see refs. 27–30). They satisfy a set of constraint functional equations, originating from general properties of unitarity, analyticity, relativistic invariance and locality. The important fact is that these functional equations do not refer to a specific operator $\mathcal{O}$, the S-matrix is the only dynamical information needed. The nature of the operator is reflected only via normalization constants for the form factors. In what follows, under $\mathcal{O}_a$ we will understand an exponential operator

$$\mathcal{O}_a(\boldsymbol{r}) = \exp(\mathrm{i}a\phi(\boldsymbol{r})) \tag{3.3}$$

The form-factor representation of two-point correlation functions (3.1) is particularly useful for large distance $|\boldsymbol{r}-\boldsymbol{r}'|$. In the limit $|\boldsymbol{r}-\boldsymbol{r}'| \to \infty$, the dominant contribution to the truncated function $\langle \mathcal{O}_a(\boldsymbol{r})\, \mathcal{O}_{a'}(\boldsymbol{r}')\rangle_{\mathrm{T}} = \langle \mathcal{O}_a(\boldsymbol{r})\, \mathcal{O}_{a'}(\boldsymbol{r}')\rangle - \langle \mathcal{O}_a\rangle\langle \mathcal{O}_{a'}\rangle$ comes from a multi-particle intermediate state with the minimum value of the total particle mass $\sum_{j=1}^N m_{\epsilon_j}$, at the point of vanishing rapidities $\theta_j \to 0$. The corresponding exponential decay $\exp(-|\boldsymbol{r}-\boldsymbol{r}'| \sum_{j=1}^N m_{\epsilon_j})$ is modified by a slower (inverse power law) decaying function which particular form depends on the form factors. For the sine-Gordon theory (2.3) (or, equivalently, the 2D TCP), in the region $0 < b^2 < 1/2$ $(0 < \beta < 2)$, the lightest particle in the spectrum is the elementary breather B_1. As is clear from (2.12), approaching $b^2 \to 1/2$ resp. $\xi \to 1$ $(\beta \to 2)$, its mass $m_1 \to 2M$. Note that, according to Eq. (2.16), M is finite at $b^2 = 1/2$, resp. $\xi = 1$ for a fixed value of z. The elementary breather B_1 disappears at the free-fermion point $b^2 = 1/2$ (collapse point $\beta = 2$). The only existing particles remain the topologically neutral soliton-antisoliton pair (A_+, A_-), which has just the same total mass $2M$. Thus, the total mass remains $2M$ (which is temperature-dependent) in the whole region $1/2 \leqslant b^2 < 1$ $(2 \leqslant \beta < 4)$.

3.1. $0 < b^2 < 1/2$ $(0 < \beta < 2)$

For the elementary breather B_1, the form factors $F_a(\theta_1,\ldots,\theta_N)_{1\cdots 1}$ and ${}^{1\cdots 1}F_{a'}(\theta_N,\ldots,\theta_1) = F_{a'}(\theta_N,\ldots,\theta_1)_{1\cdots 1}$ are presented for the exponential operator (3.3) in ref. 30. In the notation (3.2), they read

$$\langle 0|\, \mathrm{e}^{\mathrm{i}a\phi}\, |B_1(\theta)\rangle = -\mathrm{i}\langle \mathrm{e}^{\mathrm{i}a\phi}\rangle (\pi\lambda)^{1/2} \frac{\sin(\pi\xi a/b)}{\sin(\pi\xi)} \tag{3.4a}$$

$$\langle 0|\, \mathrm{e}^{\mathrm{i}a\phi}\, |B_1(\theta_2), B_1(\theta_1)\rangle = -\langle \mathrm{e}^{\mathrm{i}a\phi}\rangle (\pi\lambda) \left[\frac{\sin(\pi\xi a/b)}{\sin(\pi\xi)}\right]^2 R(\theta_1-\theta_2) \tag{3.4b}$$

etc., where

$$\lambda = \frac{4}{\pi}\sin(\pi\xi)\cos\left(\frac{\pi\xi}{2}\right)\exp\left\{-\int_0^{\pi\xi}\frac{\mathrm{d}t}{\pi}\frac{t}{\sin t}\right\} \tag{3.5}$$

and the function $R(\theta)$ is represented in the range $-2\pi+\pi\xi < \mathrm{Im}(\theta) < -\pi\xi$ by the integral

$$R(\theta) = \mathcal{N} \exp\left\{8 \int_0^\infty \frac{\mathrm{d}t}{t} \frac{\sinh t \sinh(t\xi) \sinh(t(1+\xi))}{\sinh^2(2t)} \sinh^2 t \left(1 - \frac{\mathrm{i}\theta}{\pi}\right)\right\} \tag{3.6a}$$

$$\mathcal{N} = \exp\left\{4 \int_0^\infty \frac{\mathrm{d}t}{t} \frac{\sinh t \sinh(t\xi) \sinh(t(1+\xi))}{\sinh^2(2t)}\right\} \tag{3.6b}$$

An analytic continuation of this representation to the case of interest $\mathrm{Im}(\theta) = 0$ is given by the relation

$$R(\theta)\, R(\theta \pm \mathrm{i}\pi) = \frac{\sinh\theta}{\sinh\theta \mp \mathrm{i}\sin(\pi\xi)} \tag{3.7}$$

Note a different notation in comparison with ref. 30.

The pair correlation function $h_{\sigma\sigma'}(\boldsymbol{r}, \boldsymbol{r}')$ (2.7) is expressible as follows

$$h_{\sigma\sigma'}(\boldsymbol{r}, \boldsymbol{r}') = \frac{\langle \mathrm{e}^{\mathrm{i}\sigma b\phi(\boldsymbol{r})} \mathrm{e}^{\mathrm{i}\sigma' b\phi(\boldsymbol{r}')} \rangle_{\mathrm{T}}}{\langle \mathrm{e}^{\mathrm{i}\sigma b\phi} \rangle \langle \mathrm{e}^{\mathrm{i}\sigma' b\phi} \rangle} \tag{3.8}$$

Using the form-factor representation (3.1), the dominant contribution to $h_{\sigma\sigma'}(\boldsymbol{r}, \boldsymbol{r}')$ in the limit $|\boldsymbol{r}-\boldsymbol{r}'| \to \infty$ is given by the one-breather B_1 state with mass m_1 and the form factor (3.4a). Then, since

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\theta}{2}\, \mathrm{e}^{-rm_1 \cosh\theta} = K_0(m_1 r) \sim \left(\frac{\pi}{2m_1 r}\right)^{1/2} \exp(-m_1 r) \tag{3.9}$$

where K_0 is the modified Bessel function of second kind,[36] one finds, at asymptotically large distance, that

$$h_{\sigma\sigma'}(r) \sim \sigma\sigma' h(r) \qquad \text{as} \quad r \to \infty \tag{3.10}$$

with

$$h(r) = -\lambda \left(\frac{\pi}{2m_1 r}\right)^{1/2} \exp(-m_1 r) \tag{3.11}$$

Using the thermodynamic formulae derived in Section 2, the lightest-breather mass m_1 is expressed as follows

$$m_1 = \kappa \left[\frac{\sin(\pi\beta/(4-\beta))}{\pi\beta/(4-\beta)} \right]^{1/2} \tag{3.12a}$$

$$= \kappa \left[1 - \frac{\pi^2}{192} \beta^2 - \frac{\pi^2}{384} \beta^3 + O(\beta^4) \right] \tag{3.12b}$$

where

$$\kappa = (2\pi\beta n)^{1/2} \tag{3.13}$$

denotes the inverse Debye length. The parameter λ (3.5) takes the form

$$\lambda = \frac{4}{\pi} \sin\left(\frac{\pi\beta}{4-\beta}\right) \cos\left(\frac{\pi\beta}{2(4-\beta)}\right) \exp\left\{ -\int_0^{\frac{\pi\beta}{4-\beta}} \frac{\mathrm{d}t}{\pi} \frac{t}{\sin t} \right\} \tag{3.14a}$$

$$= \beta \left[1 - \left(\frac{1}{32} + \frac{7\pi^2}{384}\right) \beta^2 - \left(\frac{1}{96} + \frac{23\pi^2}{2304}\right) \beta^3 + O(\beta^4) \right] \tag{3.14b}$$

The specific dependence of $h_{\sigma\sigma'}(r)$ on the product of charges $\sigma\sigma'$, formula (3.10), means that the two-particle correlations are determined at large distances by the charge-charge correlation function. Indeed,

$$\frac{1}{4} \sum_{\sigma, \sigma' = \pm} \sigma\sigma' h_{\sigma\sigma'}(r) = h(r) \tag{3.15}$$

On the other hand, the density-density correlation function $\sum_{\sigma, \sigma'=\pm} h_{\sigma\sigma'}(r)/4$ vanishes for the lowest one-breather B_1 state, and becomes nonzero only for the next two-breather B_1 state, with the form factor (3.4b) and much faster exponential decay $\exp(-2m_1 r)$.

3.2. $1/2 \leqslant b^2 < 1$ $(2 \leqslant \beta < 4)$

The form factors for the soliton-antisoliton pair with topological $U(1)$-charges $\epsilon = \pm$ are nonvanishing only for $U(1)$ neutral states $\sum_{j=1}^N \epsilon_j = 0$, with N being inevitably an even number. Simultaneously, ${}^{\epsilon_N \cdots \epsilon_1} F_{a'}(\theta_N, \ldots, \theta_1) = F_{a'}(\theta_N, \ldots, \theta_1)_{\epsilon_N^* \cdots \epsilon_1^*}$ where $\epsilon^* = -\epsilon$. Form factors with different assignments of the charges $\{\epsilon_i\}$ are related by the relation[24]

$$F_a(\theta_1, \ldots, \theta_i, \theta_{i+1}, \ldots \theta_N)_{\epsilon_1 \cdots \epsilon_i \epsilon_{i+1} \cdots \epsilon_N} = -F_a(\theta_1, \ldots, \theta_{i+1}, \theta_i, \ldots \theta_N)_{\epsilon_1 \cdots \epsilon_{i+1} \epsilon_i \cdots \epsilon_N} \tag{3.16}$$

and the hermiticity of the operator (3.3) implies

$$F_a(\theta_1, \ldots, \theta_N)_{\epsilon_1 \cdots \epsilon_N} = F_{-a}(\theta_N, \ldots, \theta_1)_{\epsilon_N^* \cdots \epsilon_1^*} \tag{3.17}$$

In the notation (3.2), the two-particle form factors for the exponential operator (3.3) with the special value of $a = b$ read[30]

$$\langle 0|\, e^{ib\phi}\, |A_{\pm}(\theta_2), A_{\mp}(\theta_1)\rangle = \langle e^{ib\phi}\rangle\, G^{b}_{\mp\pm}(\theta_1 - \theta_2) \tag{3.18}$$

where

$$G^{b}_{\mp\pm}(\theta) = -2G(\theta)\sinh(\theta)\,\mathrm{cotg}\left(\frac{\pi\xi}{2}\right)\exp\left(\mp\frac{\theta+i\pi}{2\xi}\right)\left[\xi\sinh\left(\frac{\theta+i\pi}{\xi}\right)\right]^{-1} \tag{3.19a}$$

$$G(\theta) = \exp\left\{\int_0^\infty \frac{dt}{t}\,\frac{\sinh(t(\xi-1))}{\sinh(2t)\cosh t\,\sinh(t\xi)}\sinh^2 t\left(1-\frac{i\theta}{\pi}\right)\right\} \tag{3.19b}$$

Note a different notation. The form factors of the operator $\exp(-ib\phi)$ are deducible from (3.18) and (3.19) by using relation (3.17).

In the region $2 \leqslant \beta < 4$, for fixed z, the total particle density n is infinite, so that only the fugacity z can be chosen as a legitimate parameter. Instead of h, which vanishes in this region, the Ursell function (2.8) is considered. Using (3.16)–(3.19) in the form-factor representation (3.1) given by the contribution of the soliton-antisoliton pair, we get, as previously,

$$U_{\sigma\sigma'}(r) \sim \sigma\sigma' U(r) \qquad \text{as} \quad r \to \infty \tag{3.20}$$

where $U(r)$ is the leading asymptotic $r \to \infty$ term of the integral

$$-\frac{M^4\cos(\pi/\xi)}{16b^4}\int_{-\infty}^{\infty}\frac{d\theta_1\,d\theta_2}{(2\pi)^2}\,G(\theta)\,G(-\theta)\sinh^2\theta \times\left[\sinh\left(\frac{\theta+i\pi}{\xi}\right)\sinh\left(\frac{-\theta+i\pi}{\xi}\right)\right]^{-1} e^{-Mr(\cosh\theta_1+\cosh\theta_2)} \tag{3.21}$$

with $\theta = \theta_1 - \theta_2$.

At $\beta = 2$, since $\xi = 1$ it holds $G(\theta) = 1$, and therefore

$$U(r) = -\frac{m^3}{8\pi r}\exp(-2mr), \qquad \beta = 2 \tag{3.22a}$$

where

$$m = 2\pi z \tag{3.22b}$$

is the soliton mass M at $\beta = 2$, see formula (2.16) with $b^2 = 1/2$ and $\xi = 1$.

For $\beta > 2$, it can be readily shown from (3.21) that

$$U(r) \propto -\frac{1}{r^2} \exp(-2Mr), \qquad 2 < \beta < 4 \tag{3.23a}$$

where the soliton mass M is given by formula (2.16) in the following form

$$M = \frac{2\Gamma(\beta/[2(4-\beta)])}{\sqrt{\pi}\,\Gamma(2/(4-\beta))} \left[\frac{\pi z \Gamma(1-\beta/4)}{\Gamma(\beta/4)} \right]^{\frac{1}{2-\beta/2}} \tag{3.23b}$$

The prefactor in (3.23a) could, in principle, be obtained from (3.21), but we omit this complicated calculation.

4. ANALYTIC CHECKS OF THE RESULTS

4.1. Small Coupling Expansion

In this section, the small-β expansion (3.12) and (3.14) of the asymptotic form (3.11) of the charge correlation function will be checked by a direct calculation using the bond-renormalized Mayer expansion in density.[3, 13]

The two-dimensional Fourier transforms which will be needed are defined here, for instance for the charge correlation function $h(r)$, as

$$\hat{h}(k) = \int_0^\infty h(r)\, J_0(kr)\, r\, \mathrm{d}r \tag{4.1a}$$

$$h(r) = \int_0^\infty \hat{h}(k)\, J_0(kr)\, k\, \mathrm{d}k \tag{4.1b}$$

where J_0 is a Bessel function. $\hat{h}(k)$ is related to the Fourier transform $\hat{c}(k)$ of the charge direct correlation function by the Ornstein–Zernike relation

$$\hat{h}(k) = \frac{\hat{c}(k)}{1 - 2\pi n \hat{c}(k)} \tag{4.2}$$

where n is the total number density of the particles (in ref. 13, h and c were called h' and c', respectively).

At the lowest order in β (Debye–Hückel approximation), $c(r) = \beta \ln r$, $\hat{c}(k) = -\beta/k^2$, and

$$\hat{h}(k) = -\frac{\beta}{k^2 + \kappa^2} \tag{4.3}$$

where κ is the inverse Debye length defined in Eq. (3.13). Thus,

$$h(r) = -\beta K_0(\kappa r) \tag{4.4}$$

where K_0 is a modified Bessel function. The asymptotic form of $h(r)$ is[36]

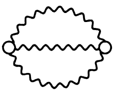
$$h(r) = -\beta \left(\frac{\pi}{2\kappa r}\right)^{1/2} \exp(-\kappa r) \tag{4.5}$$

in agreement with (3.11)–(3.14).

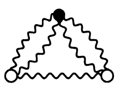
Further terms in the β-expansion of $c(r)$ are given by the renormalized Mayer expansion in density. Graphs of the excess Helmholtz free energy contributing to $c(r)$ are the ones which have their two root-vertices with an odd bond-coordination and their field vertices with an even bond-coordination. Up to order β^4, for a fixed value of κ,

$$c(r) = \beta \ln r + \beta^3 c_3(r) + \beta^4 c_4(r) \tag{4.6}$$

where $\beta^3 c_3$ and $\beta^4 c_4$ are defined by the diagrams [where a wavy line represents the renormalized bond $-\beta K_0(\kappa r)$]:

$\beta^3 c_3(r) =$  $= -\dfrac{\beta^3}{6} K_0^3(\kappa r)$ (4.7a)

and

$\beta^4 c_4(r) =$  $= -\dfrac{\beta^5}{4} K_0(\kappa r)\, n\kappa^{-2} \int d^2(\kappa r')\, K_0^2(\kappa r')\, K_0^2(\kappa\, |\mathbf{r} - \mathbf{r}'|)$ (4.7b)

These diagrams are of order β^3 and β^4, respectively [this is apparent for $c_4(r)$ when the factor $n\kappa^{-2}$ in front of the integral is replaced by $1/(2\pi\beta)$]. $c(r)$ has no term of order β^2. The Fourier transforms of $c_3(r)$ and $c_4(r)$ will now be shown to have convenient integral representations. From now on, κ^{-1} will be taken as the unit of length.

In the Fourier transform

$$\hat{c}_3(k) = -\frac{1}{6} \int_0^\infty K_0^3(r)\, J_0(kr)\, r\, dr \tag{4.8}$$

the integral can be viewed as $1/(2\pi)$ times the scalar product of $K_0^2(r)$ and $K_0(r)\, J_0(kr)$, which can be written as the scalar product of their Fourier transforms. These transforms are[3, 36]

$$\hat{G}(l) = \int_0^\infty K_0^2(r)\, J_0(lr)\, r\, \mathrm{d}r = \frac{\ln\left[\frac{l}{2} + \sqrt{1+\left(\frac{l}{2}\right)^2}\right]}{l\sqrt{1+\left(\frac{l}{2}\right)^2}} \tag{4.9}$$

and

$$\hat{H}_k(l) = \int_0^\infty K_0(r)\, J_0(kr)\, J_0(lr)\, r\, \mathrm{d}r = (1+k^4+l^4-2k^2l^2+2k^2+2l^2)^{-1/2} \tag{4.10}$$

Therefore, setting $l = 2\sinh\varphi$, one finds

$$\begin{aligned} \hat{c}_3(k) &= -\frac{1}{6}\int_0^\infty \hat{G}(l)\, \hat{H}_k(l)\, l\, \mathrm{d}l \\ &= -\frac{1}{3}\int_0^\infty \frac{\mathrm{d}\varphi\, \varphi}{[16\sinh^4\varphi + 16\sinh^2\varphi - 8(k^2+1)\sinh^2\varphi + (k^2+1)^2]^{1/2}} \end{aligned} \tag{4.11}$$

In the expression (4.7b) of $c_4(r)$, there is an integral $I(r)$, which is the convolution of K_0^2 with itself. Its Fourier transform is $\hat{I}(l) = 2\pi[\hat{G}(l)]^2$. Again, in

$$\hat{c}_4(k) = -\frac{1}{8\pi}\int_0^\infty K_0(r)\, I(r)\, J_0(kr)\, r\, \mathrm{d}r \tag{4.12}$$

the integral can be viewed as $1/(2\pi)$ times a scalar product, now of $I(r)$ and $K_0(r)\, J_0(kr)$. As above, in terms of the Fourier transforms $\hat{I}(l)$ and $\hat{H}_k(l)$,

$$\hat{c}_4(k) = -\frac{1}{8\pi}\int_0^\infty \hat{I}(l)\, \hat{H}_k(l)\, l\, \mathrm{d}l = -\frac{1}{4}\int_0^\infty [\hat{G}(l)]^2\, \hat{H}_k(l)\, l\, \mathrm{d}l \tag{4.13}$$

Using (4.9) and (4.10) in (4.13), and again setting $l = 2\sinh\varphi$, one finds

$$\hat{c}_4(k) = -\frac{1}{4}\int_0^\infty \frac{\mathrm{d}\varphi\, \varphi^2}{\left[\begin{array}{l}\sinh\varphi\cosh\varphi[16\sinh^4\varphi \\ \quad +16\sinh^2\varphi - 8(k^2+1)\sinh^2\varphi + (k^2+1)^2]^{1/2}\end{array}\right]} \tag{4.14}$$

Finally, up to order β^4,

$$\hat{c}(k) = -\frac{\beta}{k^2}[1-\beta^2 k^2 \hat{c}_3(k) - \beta^3 k^2 \hat{c}_4(k)] \tag{4.15}$$

where $\hat{c}_3$ and $\hat{c}_4$ are given by (4.11) and (4.14).

Using (4.15) in (4.2) (with $\kappa = 1$) gives

$$\hat{h}(k) = -\frac{\beta[1-\beta^2 k^2 \hat{c}_3(k) - \beta^3 k^2 \hat{c}_4(k)]}{k^2+1-\beta^2 k^2 \hat{c}_3(k) - \beta^3 k^2 \hat{c}_4(k)} \tag{4.16}$$

We are interested in the asymptotic behavior of $h(r)$, which is governed by the poles of $\hat{h}(k)$ closest to the real axis. When $\beta \to 0$, these poles are at $k = \pm \mathrm{i}$, i.e., at $k^2 = -1$. For finding the location and the contributions of these poles in a β expansion when the β^2 and β^3 terms in the denominator of (4.16) are taken into account, it is enough to expand $\hat{c}_3$ and $\hat{c}_4$ around $k^2 = -1$ up to first order in $1+k^2$.

However, some care is needed for deriving this expansion from the expressions (4.11) and (4.14). Indeed, these expressions were obtained by using the Fourier transform (4.10) of $K_0(r)\, J_0(kr)$. However, for complex k, this integral diverges at large r when $|\mathrm{Im}\, k| > 1$. Thus, the integral representations (4.11) and (4.14) are valid for $1+k^2$ real positive but are expected to exhibit some singularity at $1+k^2=0$ [although the original function $\hat{c}_3(k)$ defined by (4.8) and its counterpart for $\hat{c}_4(k)$ (4.12) are regular around $1+k^2=0$; for instance, (4.8) defines a function of k which is analytical in the whole strip $|\mathrm{Im}\, k| < 3$, thus in particular for any real value of $1+k^2 > -8$]. The singularity in (4.11) or (4.14) can be seen directly if a naive expansion with respect to $x = 1+k^2$ is attempted: the coefficient of x^2 is an integral which diverges at small φ. This is a warning that one must be careful when computing the coefficient of the previous term (of order x) in the expansion, as follows.

The zeros of the denominator in (4.11) are $\sinh^2 \varphi = -(1/2)[1-(x/2)/\pm(1-x)^{1/2}]$, i.e., for small x,

$$\hat{c}_3(k) = -\frac{1}{12}\int_0^\infty \frac{\mathrm{d}\varphi\, \varphi}{[\sinh^2\varphi + (x^2/16) + \cdots]^{1/2}\,[\cosh^2\varphi - (x/2) + \cdots]^{1/2}} \tag{4.17}$$

While the second factor in the denominator can be expanded with respect to x, the first factor is the dangerous one in which the x^2 term will give a contribution of first order in x and should not be suppressed. To first order in x,

$$\hat{c}_3(k) = -\frac{1}{12}\int_0^\infty \frac{d\varphi\, \varphi}{\cosh\varphi[\sinh^2\varphi + (x^2/16)]^{1/2}}\left[1 + \frac{x}{4\cosh^2\varphi}\right]$$
$$= -\frac{1}{12}\int_0^\infty \frac{d\varphi\, \sinh\varphi}{\cosh\varphi[\sinh^2\varphi + (x^2/16)]^{1/2}}$$
$$+\frac{1}{12}\int_0^\infty \frac{(\sinh\varphi - \varphi)\, d\varphi}{\cosh\varphi[\sinh^2\varphi + (x^2/16)]^{1/2}}$$
$$-\frac{x}{48}\int_0^\infty \frac{\varphi\, d\varphi}{\cosh^3\varphi[\sinh^2\varphi + (x^2/16)]^{1/2}} \tag{4.18}$$

In an expansion of $\hat{c}_3$ to first order in x, the third integral in the rhs of (4.18) is needed only for $x^2 = 0$. By an integration per partes, it is found to be

$$\int_0^\infty \frac{d\varphi\, \varphi}{\cosh^3\varphi\, \sinh\varphi} = \frac{\pi^2}{8} - \frac{1}{2} \tag{4.19}$$

A naive expansion of the second integral in the rhs of (4.18) gives a finite coefficient for the x^2 term. Thus, the integral has no term of first order in x and can be evaluated at $x^2 = 0$. Using tabulated integrals,[36] one finds

$$\int_0^\infty \frac{d\varphi(\sinh\varphi - \varphi)}{\cosh\varphi\, \sinh\varphi} = \frac{\pi}{2} - \frac{\pi^2}{8} \tag{4.20}$$

Finally, the first integral in the rhs of (4.18), which is the delicate one, can be exactly evaluated, by taking $\cosh\varphi$ as the integration variable. The result does have a term of order x, which would have been missed if x^2 had been neglected in the integrand:

$$\int_0^\infty \frac{d\varphi\, \sinh\varphi}{\cosh\varphi[\sinh^2\varphi + (x^2/16)]^{1/2}} = \frac{\pi}{2} - \frac{x}{4} + \cdots \tag{4.21}$$

It should be noted that (4.21) has been derived for x real positive. For x real negative, x must be replaced by $|x|$ in the rhs of (4.21), which confirms that the integral representation (4.11) has a singularity (a kink) at $x = 0$. However, since the original expression (4.8) is regular around $x = 0$ and the integral representation (4.11) is valid for x real positive, (4.21) can be used for obtaining the correct analytical behavior of $\hat{c}_3(k)$ near $1+k^2=0$. Using (4.19), (4.20), and (4.21) in (4.18) gives the $1+k^2$ expansion

$$\hat{c}_3(k) = -\frac{\pi^2}{96} + \left(\frac{1}{32} - \frac{\pi^2}{384}\right)(1+k^2) + \cdots \tag{4.22}$$

A similar approach can be used for expanding the expression (4.14) of $\hat{c}_4(k)$. To first order in $x = 1+k^2$,

$$\begin{aligned}\hat{c}_4(k) = &-\frac{1}{16}\int_0^\infty \frac{d\varphi \sinh\varphi}{\cosh^2\varphi[\sinh^2\varphi + (x^2/16)]^{1/2}} \\ &+ \frac{1}{16}\int_0^\infty \frac{d\varphi(\sinh^2\varphi - \varphi^2)}{\sinh^2\varphi\cosh^2\varphi} - \frac{x}{64}\int_0^\infty \frac{d\varphi\,\varphi^2}{\sinh^2\varphi\cosh^4\varphi}\end{aligned} \tag{4.23}$$

By simple manipulations, the third integral in the rhs of (4.23) is found to be

$$\int_0^\infty \frac{d\varphi\,\varphi^2}{\sinh^2\varphi\cosh^4\varphi} = \frac{\pi^2}{36} + \frac{1}{3} \tag{4.24}$$

From tables,[36] the second integral is

$$\int_0^\infty \frac{d\varphi(\sinh^2\varphi - \varphi^2)}{\sinh^2\varphi\cosh^2\varphi} = 1 - \frac{\pi^2}{12} \tag{4.25}$$

Finally, the first integral can be exactly evaluated and afterwards expanded in x with the result

$$\int_0^\infty \frac{d\varphi \sinh\varphi}{\cosh^2\varphi[\sinh^2\varphi + (x^2/16)]^{1/2}} = 1 - \frac{x}{4} + \cdots \tag{4.26}$$

Using (4.24), (4.25), and (4.26) in (4.23) gives

$$\hat{c}_4(k) = -\frac{\pi^2}{192} + \left(\frac{1}{96} - \frac{\pi^2}{2304}\right)(1+k^2) + \cdots \tag{4.27}$$

By using (4.22) and (4.27) in the denominator of (4.16), it is seen that, at order β^3, this denominator has poles at $k = \pm i m_1$ with

$$m_1 = 1 - \frac{\pi^2}{192}\beta^2 - \frac{\pi^2}{384}\beta^3 \tag{4.28}$$

The asymptotic behavior of $h(r)$ is governed by these poles and the corresponding residues, i.e., by writing $\hat{h}(k)$ in the form $-\lambda(k^2)(k^2+m_1^2)^{-1}$ and replacing k^2 by $-m_1^2$ in $\lambda(k^2)$. One obtains

$$\hat{h}(k) \sim -\frac{\lambda}{k^2+m_1^2} \tag{4.29}$$

where, at order β^3,

$$\lambda = \beta\left[1-\left(\frac{1}{32}+\frac{7\pi^2}{384}\right)\beta^2-\left(\frac{1}{96}+\frac{23\pi^2}{2304}\right)\beta^3\right] \tag{4.30}$$

The corresponding $h(r)$ is $-\lambda K_0(m_1 r)$ which has the asymptotic behavior (in units of κ^{-1})

$$h(r) \sim -\lambda\left(\frac{\pi}{2m_1 r}\right)^{1/2} \exp(-m_1 r) \tag{4.31}$$

with m_1 and λ given by (4.28) and (4.30). This is in agreement with the small-β expansions of these quantities (3.12b) and (3.14b) obtained from the exact result.

4.2. Free-Fermion $\beta=2$ Point

At the collapse point $\beta = 2$, the Ursell functions were found in refs. 15 and 17. For fixed fugacity z, they read

$$U_{\sigma,\sigma}(r) = -\left(\frac{m^2}{2\pi}\right)^2 K_0^2(mr) \tag{4.32a}$$

$$U_{\sigma,-\sigma}(r) = \left(\frac{m^2}{2\pi}\right)^2 K_1^2(mr) \tag{4.32b}$$

where $m = 2\pi z$, K_0 and K_1 are modified Bessel functions and $\sigma = \pm$. Consequently,

$$U_{\sigma\sigma'}(r) \sim -\sigma\sigma' \frac{m^3}{8\pi r} \exp(-2mr) \qquad \text{as} \quad r \to \infty \tag{4.33}$$

in full agreement with (3.20) and (3.22).

5. CONCLUSION

In this paper, we have derived the exact large-distance behavior of particle correlation functions for the 2D two-component plasma by using the form-factor theory of the equivalent sine-Gordon model. The asymptotic decay is always exponential, with a continuously varying correlation length. The correlation length is determined by the particle spectrum of the sine-Gordon model: in the stability regime of the inverse temperatures $0 < \beta < 2$, it is equal to the inverse mass $1/m_1$ of the lightest breather B_1, and in the collapse region $2 \leqslant \beta \leqslant 4$, where the breathers disappear, it is equal to the inverse total mass $1/(2M)$ of the only present particles, the soliton-antisoliton pair (A_+, A_-). On the other hand, the prefactor inverse-power-law function changes discontinuously at the collapse point $\beta = 2$: it behaves like $1/\sqrt{r}$ in the stability region $0 < \beta < 2$ [see formula (3.11)], like $1/r$ at $\beta = 2$ (3.22) and like $1/r^2$ for $2 < \beta < 4$ (3.23). The change in the large-distance behavior of correlation functions from both sides of point $\beta = 2$ is a sign of a singularity, which probably prevents the construction of an analytic $(\beta - 2)$ expansion of Ursell functions and other finite statistical quantities, although all multi-particle Ursell functions are known at the collapse point.[17]

The region $2 \leqslant \beta < 4$, in which the thermodynamic collapse occurs, deserves attention. The Ursell functions calculated in this region are equivalent to the corresponding Ursell functions of the lattice version of the 2D TCP, in the continuum limit when the lattice constant $\to 0$, as was done at $\beta = 2$ in refs. 15 and 17. The phenomenon of the Kosterlitz–Thouless phase transition[1] requires the fundamental presence of a maybe small, but nonzero, hard core attached to the charged particles. This is why we do not observe a typical K-T divergence of the correlation length $\propto \exp(c/\sqrt{4-\beta})$ as $\beta \to 4^-$. On the contrary, since the soliton mass M (3.23b) goes to infinity as $\beta \to 4^-$ for fixed z, and even for $z \sim 4 - \beta$ as was considered in the usual analysis of the renormalization flow close to the K-T transition, the correlation length goes to zero when approching $\beta = 4$. In a certain sense, likewise as $\beta = 2$ is the collapse point for the thermodynamics, $\beta = 4$ is the "collapse" point for the Ursell correlation functions of our model of pointlike particles. In the interval $0 < \beta < 2$, the introduction of a small hard core is a slight perturbation which does not change the thermodynamics and the correlation functions substantially.

A next natural step is to study inhomogeneous situations when the 2D TCP is confined by an impermeable wall of dielectric constant ϵ_W. In such situations, the asymptotic decay of pair correlation functions is direction-dependent. In the case of ideal conductor $(\epsilon_W \to \infty)$ and ideal dielectric $(\epsilon_W = 0)$ rectilinear walls, the plasma can be formulated as a boundary sine-

Gordon model with Dirichlet[7] and Neumann[8] boundary conditions, respectively. One of first attempts to determine the form factors of such integrable boundary theories was made in ref. 37.

ACKNOWLEDGMENTS

We are indebted to K. Chadan for having helped us to understand the limits of validity of the integral representation (4.11). The stay of L.Š. in LPT Orsay is supported by a NATO fellowship. A partial support by Grant VEGA 2/7174/20 is acknowledged.

REFERENCES

1. J. M. Kosterlitz and D. J. Thouless, *J. Phys. C* **6**:1181 (1973).
2. A. M. Salzberg and S. Prager, *J. Chem. Phys.* **38**:2587 (1963).
3. L. Šamaj and I. Travěnec, *J. Stat. Phys.* **101**:713 (2000).
4. C. Destri and H. de Vega, *Nucl. Phys. B* **358**:251 (1991).
5. Al. B. Zamolodchikov, *Int. J. Mod. Phys. A* **10**:1125 (1995).
6. S. Lukyanov and Al. B. Zamolodchikov, *Nucl. Phys. B* **493**:571 (1997).
7. L. Šamaj and B. Jancovici, *J. Stat. Phys.* **103**:717 (2001).
8. L. Šamaj, *J. Stat. Phys.* **103**:737 (2001).
9. S. Skorik, *Topics in 2D Integrable Field Theories with Boundary Interactions*, hep-th/9604174.
10. S. Ghoshal and A. B. Zamolodchikov, *Int. J. Mod. Phys. A* **9**:3841 (1994).
11. F. H. Stillinger and R. Lovett, *J. Chem. Phys.* **49**:1991 (1968).
12. B. Jancovici, *J. Stat. Phys.* **100**:201 (2000).
13. B. Jancovici, P. Kalinay, and L. Šamaj, *Physica A* **279**:260 (2000).
14. C. Deutsch and M. Lavaud, *Phys. Rev. A* **9**:2598 (1974).
15. F. Cornu and B. Jancovici, *J. Stat. Phys.* **49**:33 (1987).
16. M. Gaudin, *J. Phys. (France)* **46**:1027 (1985).
17. F. Cornu and B. Jancovici, *J. Chem. Phys.* **90**:2444 (1989).
18. P. J. Forrester, *J. Chem. Phys.* **95**:4545 (1991).
19. B. Jancovici and G. Téllez, *J. Stat. Phys.* **82**:609 (1996).
20. B. Jancovici and L. Šamaj, *J. Stat. Phys.* **104**:753 (2001).
21. G. Téllez, *J. Stat. Phys.* **104**:945 (2001).
22. V. S. Dotsenko and V. Fateev, *Nucl. Phys. B* **240**:312 (1984).
23. J. P. Hansen and P. Viot, *J. Stat. Phys.* **38**:823 (1985).
24. D. Bernard and A. LeClair, *Nucl. Phys. B* **426**:534 (1994); *Nucl. Phys. B* **498**:619 (1997).
25. K. Wilson, *Phys. Rev.* **179**:1499 (1969).
26. V. Fateev, D. Fradkin, S. Lukyanov, A. B. Zamolodchikov, and Al. B. Zamolodchikov, *Nucl. Phys. B* **540**:587 (1999).
27. M. Karowski and P. Weisz, *Nucl. Phys. B* **139**:455 (1978).
28. F. A. Smirnov, *Form-Factors in Completely Integrable Models of Quantum Field Theory* (World Scientific, Singapore, 1992).
29. S. Lukyanov, *Commun. Math. Phys.* **167**:183 (1995).
30. S. Lukyanov, *Mod. Phys. Lett. A* **12**:2543 (1997).

31. A. Fring, G. Mussardo, and P. Simonetti, *Nucl. Phys. B* **393** (1993) 413.
32. V. Brazhnikov and S. Lukyanov, *Nucl. Phys. B* **512**:616 (1998).
33. V. E. Korepin and N. A. Slavnov, *J. Phys. A: Math. Gen.* **31**:9283 (1998).
34. P. Minnhagen, *Rev. Mod. Phys.* **59**:1001 (1987).
35. A. B. Zamolodchikov and Al. B. Zamolodchikov, *Ann. Phys. (N.Y.)* **120**:253 (1979).
36. I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, 5th edn. (Academic Press, London, 1994).
37. B. Hou, K. Shi, Y. Wang, and W. Yang, *Int. J. Mod. Phys. A* **12**:1711 (1997).